CS 532: 3D Computer Vision
6th Set of Notes

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Lecture Outline

- Intro to Covariance Matrices
- Simultaneous Localization and Mapping
  - Based on slides by William Green (then at Drexel)
  - See also “An Introduction to the Kalman Filter” by Greg Welch and Gary Bishop
    http://www.cs.unc.edu/~welch/media/pdf/kalman_intro.pdf

Covariance

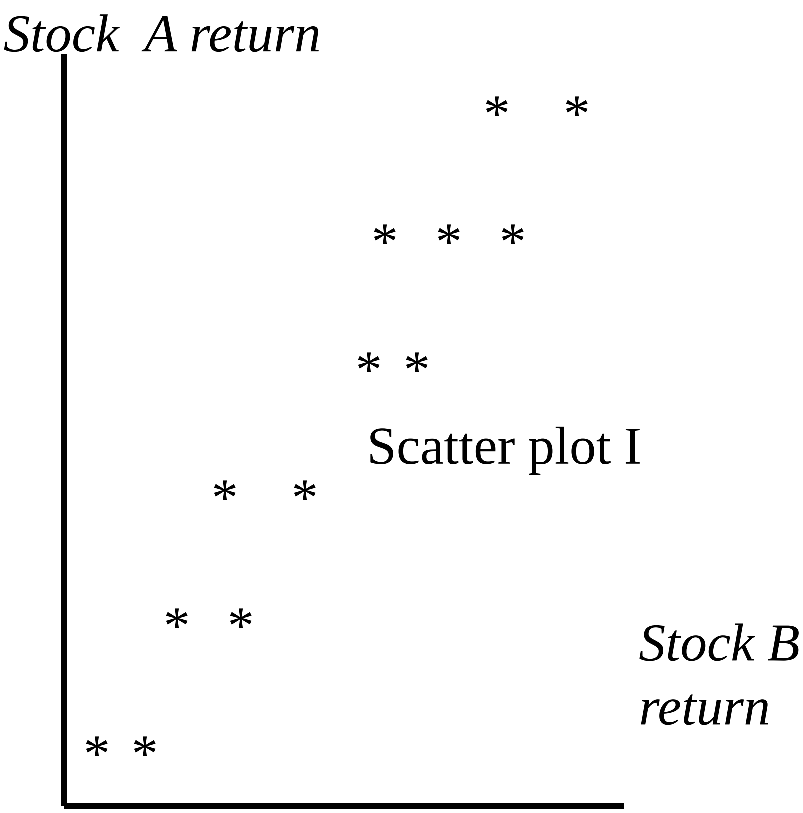
- Covariance is a numerical measure that shows how much two random variables change together

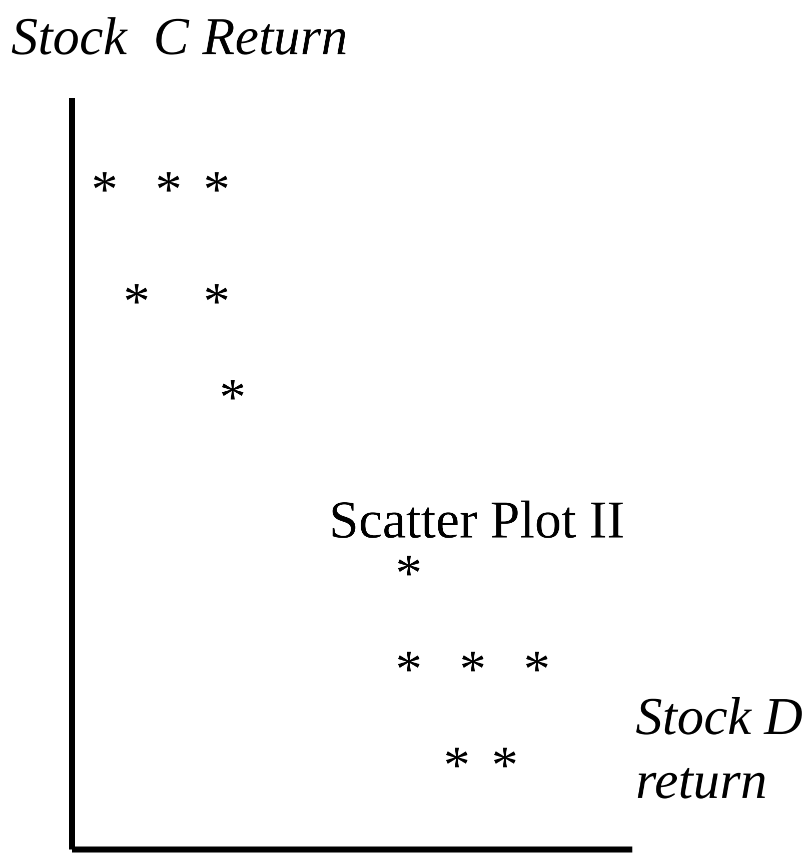
$$\sigma_{jk} = E\left[(Y_{ij} - \mu_j)(Y_{ik} - \mu_k)\right]$$

- Positive covariance: if one increases, the other is likely to increase
- Negative covariance: …
- More precisely: the covariance is a measure of the *linear* dependence between the two variables

Covariance Example

Relationships between the returns of different stocks





Correlation Coefficient

- One may be tempted to conclude that if the covariance is larger, the relationship between two variables is stronger (in the sense that they have stronger linear relationship)
- The correlation coefficient is defined as:

$$\rho_{jk} = \frac{E\left[(Y_{ij} - \mu_j)(Y_{ik} - \mu_k)\right]}{\sigma_j \sigma_k}$$

Correlation Coefficient

$$\rho_{jk} = \frac{E\left[(Y_{ij} - \mu_j)(Y_{ik} - \mu_k)\right]}{\sigma_j \sigma_k}$$

- The correlation coefficient, unlike covariance, is a measure of dependence free of scales of measurement of Y_{ij} and Y_{ik}
- By definition, correlation must take values between −1 and 1
- A correlation of 1 or −1 is obtained when there is a perfect linear relationship between the two variables

Covariance Matrix

- For the vector of repeated measures, $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{in})$, we define the covariance matrix, $\mathrm{Cov}(Y_i)$:

$$\mathrm{Cov}\begin{pmatrix} Y_{i1} \\ Y_{i2} \\ \vdots \\ Y_{in} \end{pmatrix} = \begin{pmatrix} \mathrm{Var}(Y_{i1}) & \mathrm{Cov}(Y_{i1}, Y_{i2}) & \cdots & \mathrm{Cov}(Y_{i1}, Y_{in}) \\ \mathrm{Cov}(Y_{i2}, Y_{i1}) & \mathrm{Var}(Y_{i2}) & \cdots & \mathrm{Cov}(Y_{i2}, Y_{in}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(Y_{in}, Y_{i1}) & \mathrm{Cov}(Y_{in}, Y_{i2}) & \cdots & \mathrm{Var}(Y_{in}) \end{pmatrix}$$

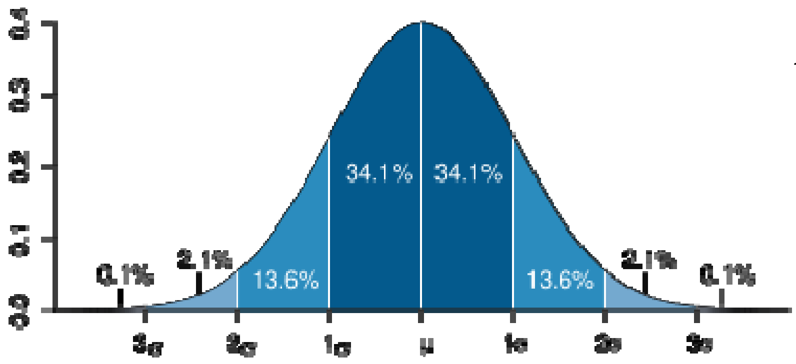
$$= \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{pmatrix},$$

where $\mathrm{Cov}\,(Y_{ij}, Y_{ik}) = \sigma_{jk} = \sigma_{kj} = \mathrm{Cov}(Y_{ik}, Y_{ij})$.

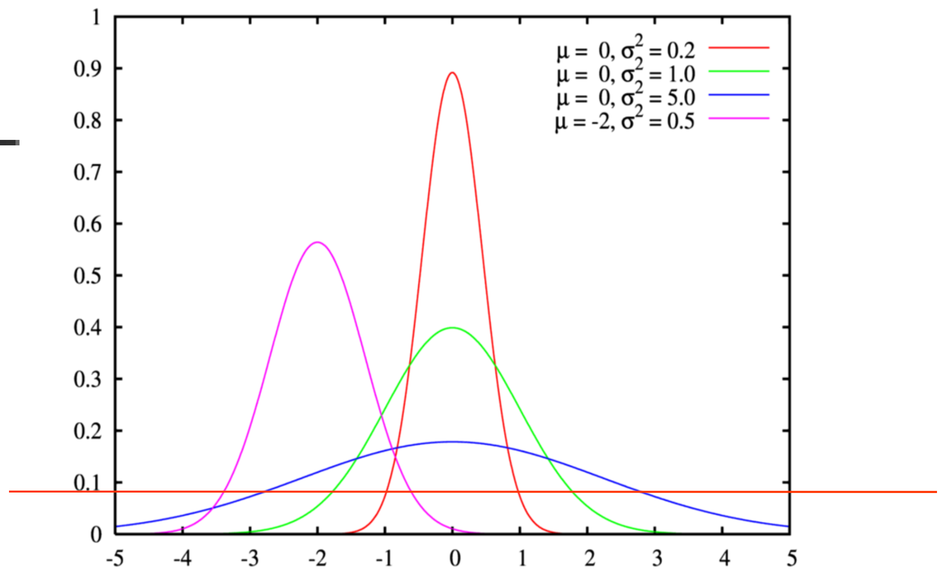
- It is a symmetric, square matrix

Variance and Confidence Intervals

- Single Gaussian (normal) random variable



$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = N(\mu, \sigma^2)$$



Multivariate Normal Density

– The multivariate normal density in d dimensions is:

$$P(x) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} \exp\left[-\frac{1}{2} (x - \mu)^t \Sigma^{-1} (x - \mu) \right]$$

where:

$x = (x_1, x_2, \ldots, x_d)^t$

$\mu = (\mu_1, \mu_2, \ldots, \mu_d)^t$ mean vector

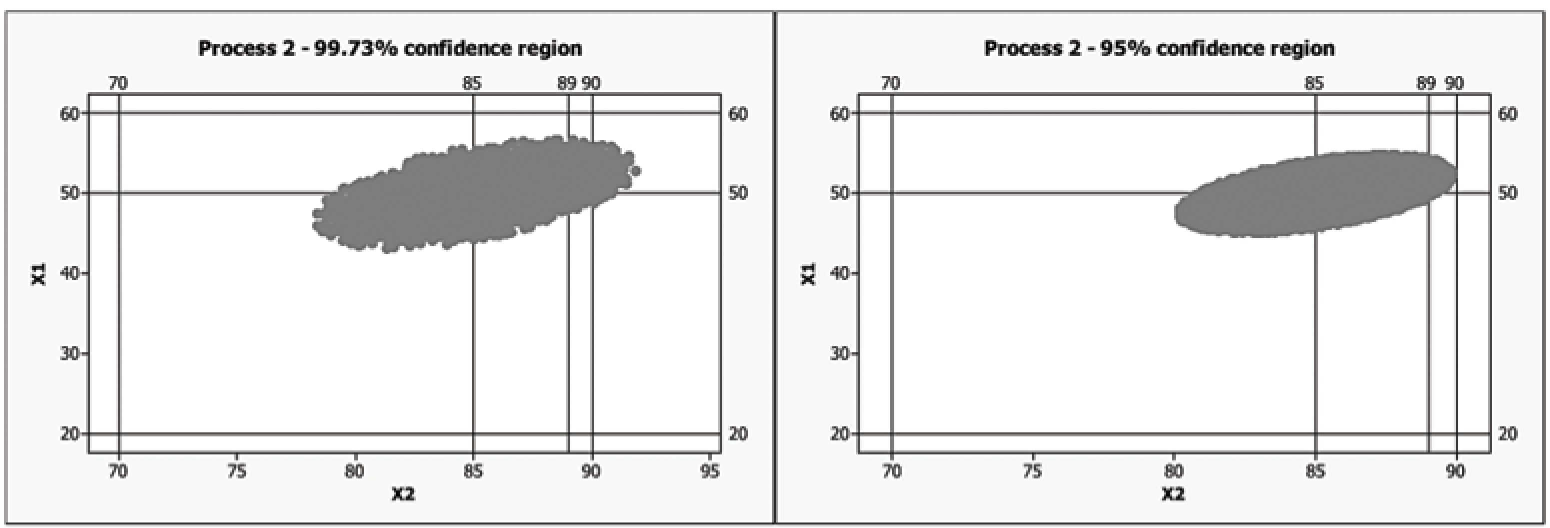
Σ = d×d covariance matrix

$|\Sigma|$ and Σ^{-1} are the determinant and inverse respectively

P(x) is larger for smaller exponents!

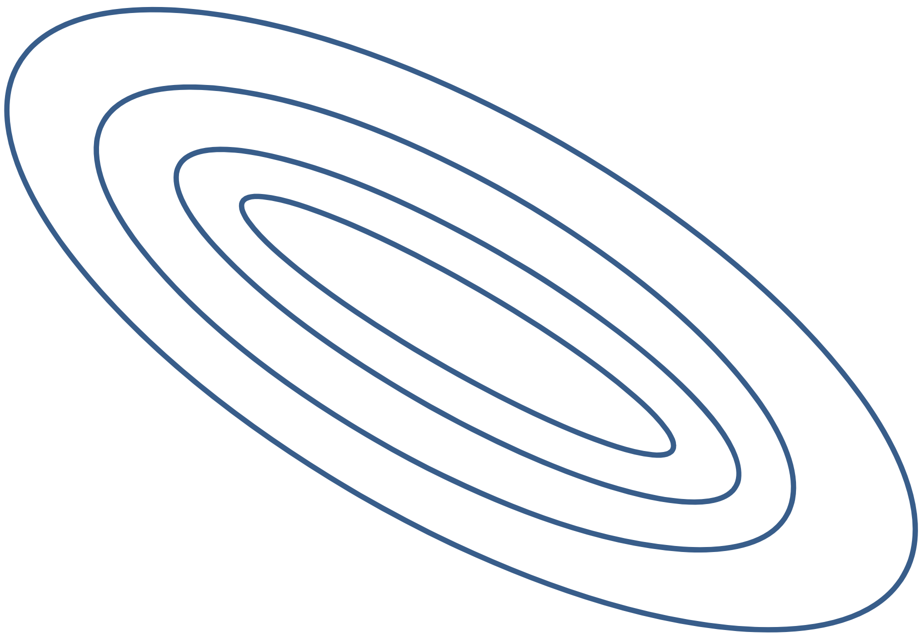
Confidence Intervals: Multi-Variate Case

- Same concept: how large is the area that contains X% of samples drawn from the distribution
- Confidence intervals are ellipsoids for normal distribution



Confidence Intervals: Multi-Variate Case

- Increasing X%, increases the size of the ellipsoids, but not their orientation and aspect ratio

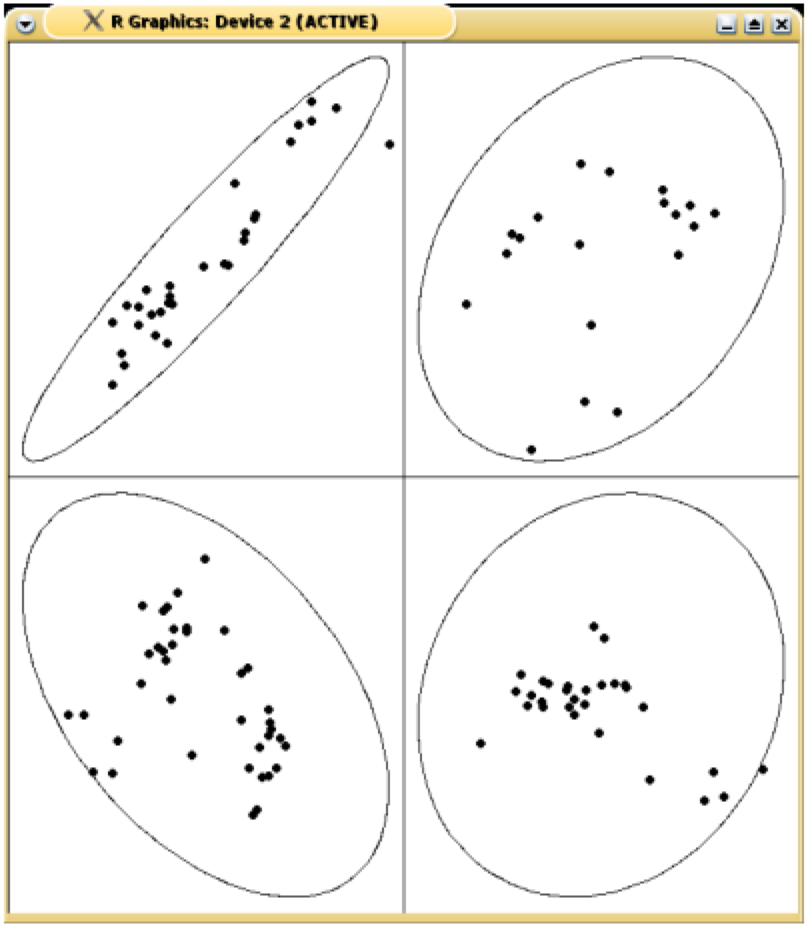


The Multi-Variate Normal Density

- Σ is positive semi definite ($x^t \Sigma x >= 0$)
  - If $x^t \Sigma x = 0$ for non-zero x then $\det(\Sigma) = 0$. This case is not interesting, p(x) is not defined
    - Two or more parameters are linearly dependent
- So we will assume Σ is positive definite ($x^t \Sigma x > 0$)
- If Σ is positive definite then so is Σ^{-1}

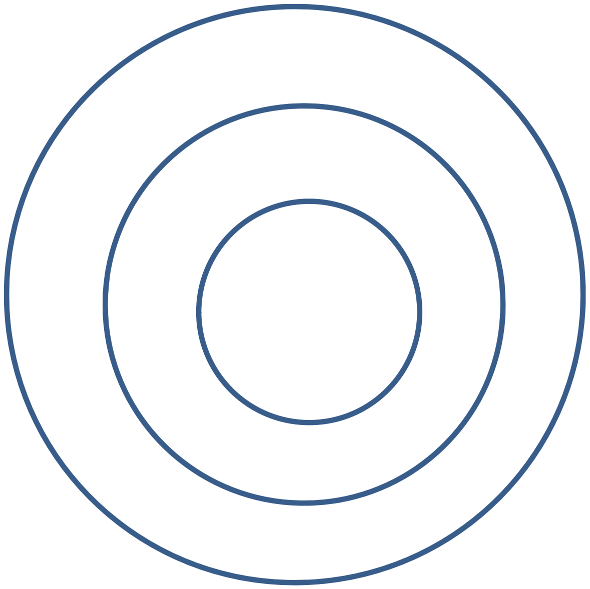
Confidence Intervals: Multi-Variate Case

- Covariance matrix determines the shape



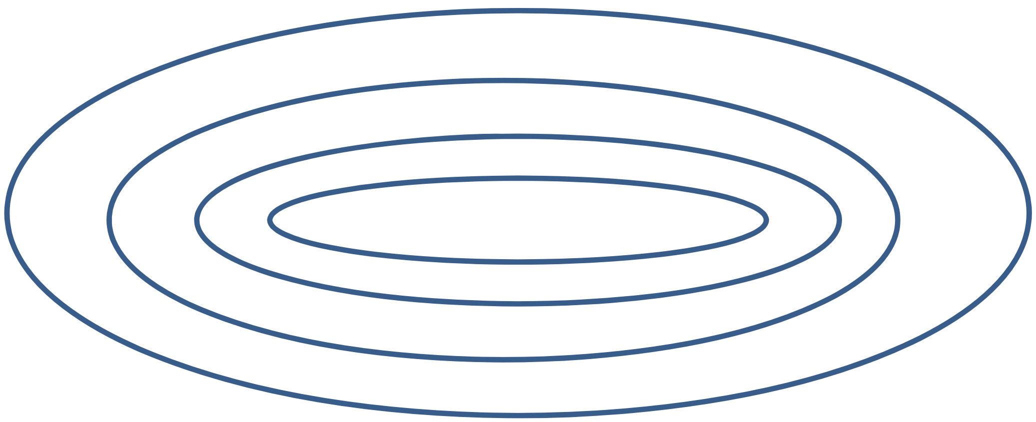
Confidence Intervals: Multi-Variate Case

- Case I: $\Sigma = \sigma^2 \mathbf{I}$
  - All variables are uncorrelated and have equal variance
- Confidence intervals are circles



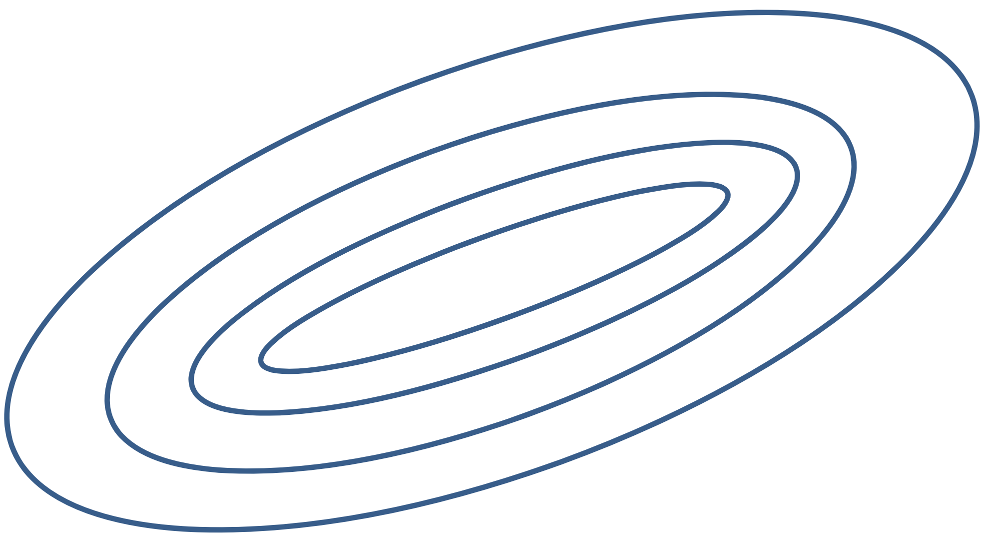
Confidence Intervals: Multi-Variate Case

- Case II: Σ diagonal, with unequal elements
  - All variables are uncorrelated but have different variances
- Confidence intervals are axis-aligned ellipsoids



Confidence Intervals: Multi-Variate Case

- Case III: Σ arbitrary
  - Variables may be correlated and have different variances
- Confidence intervals are arbitrary ellipsoids



Intro to SLAM

Introduction

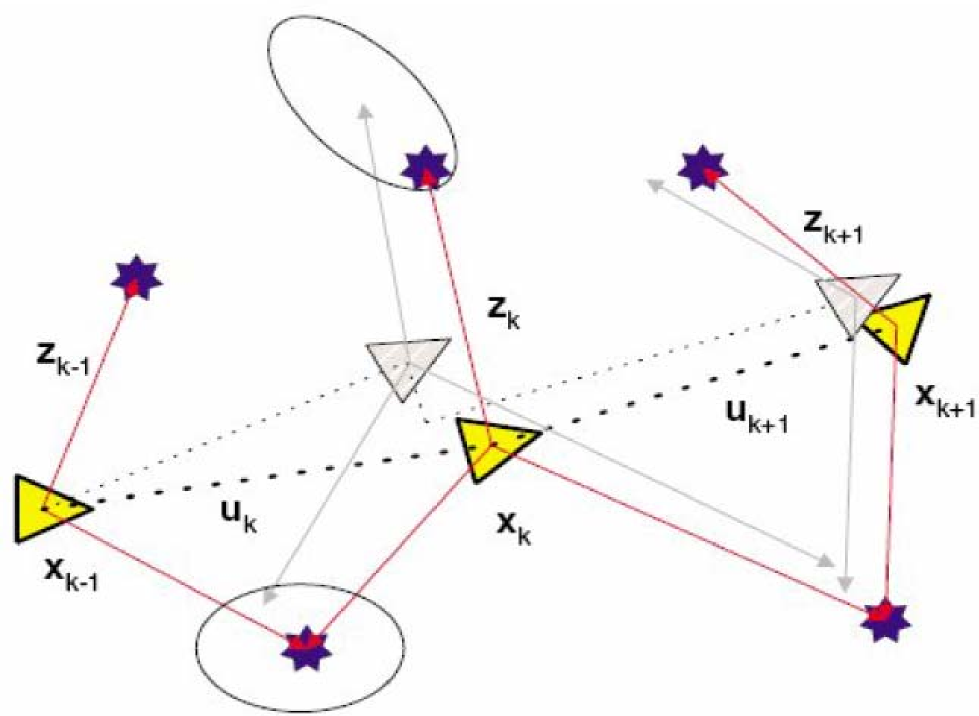
SLAM Objective

- Place a robot in an unknown location in an unknown environment and have the robot incrementally build a map of this environment while simultaneously using this map to compute vehicle location
- A solution to SLAM was seen as the "Holy Grail"
  - Would enable robots to operate in an environment without a priori knowledge of obstacle locations
- A little more than 10 years ago it was shown that a solution is possible!

The Localization Problem

- A map m of landmark locations is known a priori
- Take measurements of landmark location z_k (i.e. distance and bearing)
- Determine vehicle location x_k based on z_k
  - Need filter if sensor is noisy

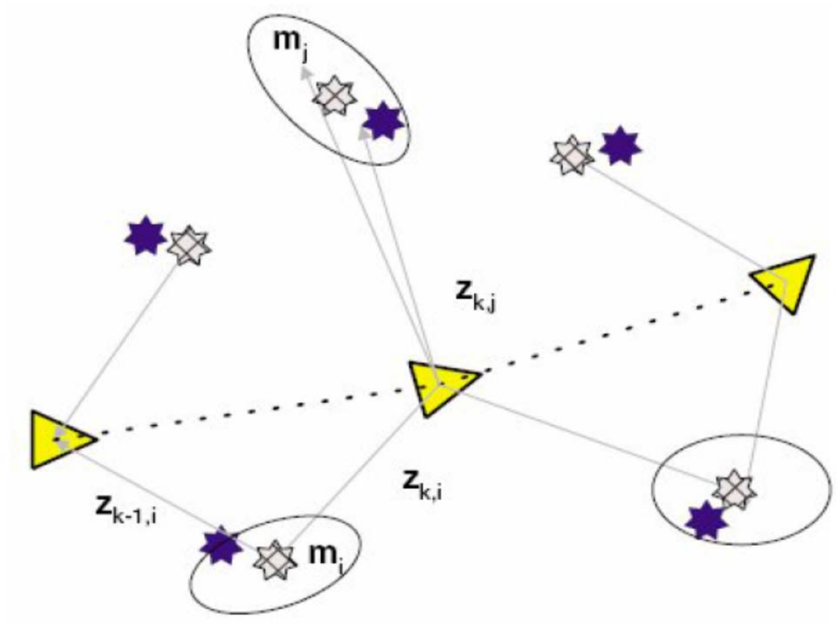
- x_k: location of vehicle at time k
- u_k: a control vector applied at k-1 to drive the vehicle from x_{k-1} to x_k
- z_k: observation of a landmark taken at time k
- X^k: history of states $\{x_1, x_2, x_3, \ldots, x_k\}$
- U^k: history of control inputs $\{u_1, u_2, u_3, \ldots, u_k\}$
- m: set of all landmarks



The Mapping Problem

- The vehicle locations X^k are provided
- Take measurements of landmark location z_k (i.e. distance and bearing)
- Build map m based on z_k
  - Need filter if sensor is noisy

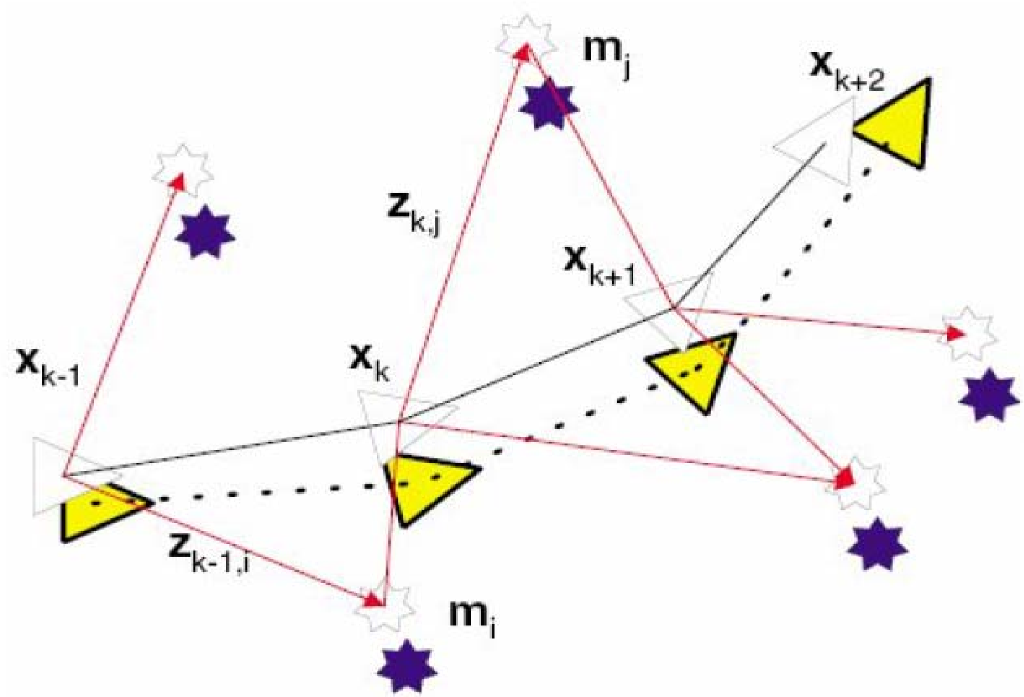
- X^k: history of states $\{x_1, x_2, x_3, \ldots, x_k\}$
- z_k: observation of a landmark taken at time k
- m_i: true location of i^{th} landmark
- m: set of all landmarks



Simultaneous Localization and Mapping

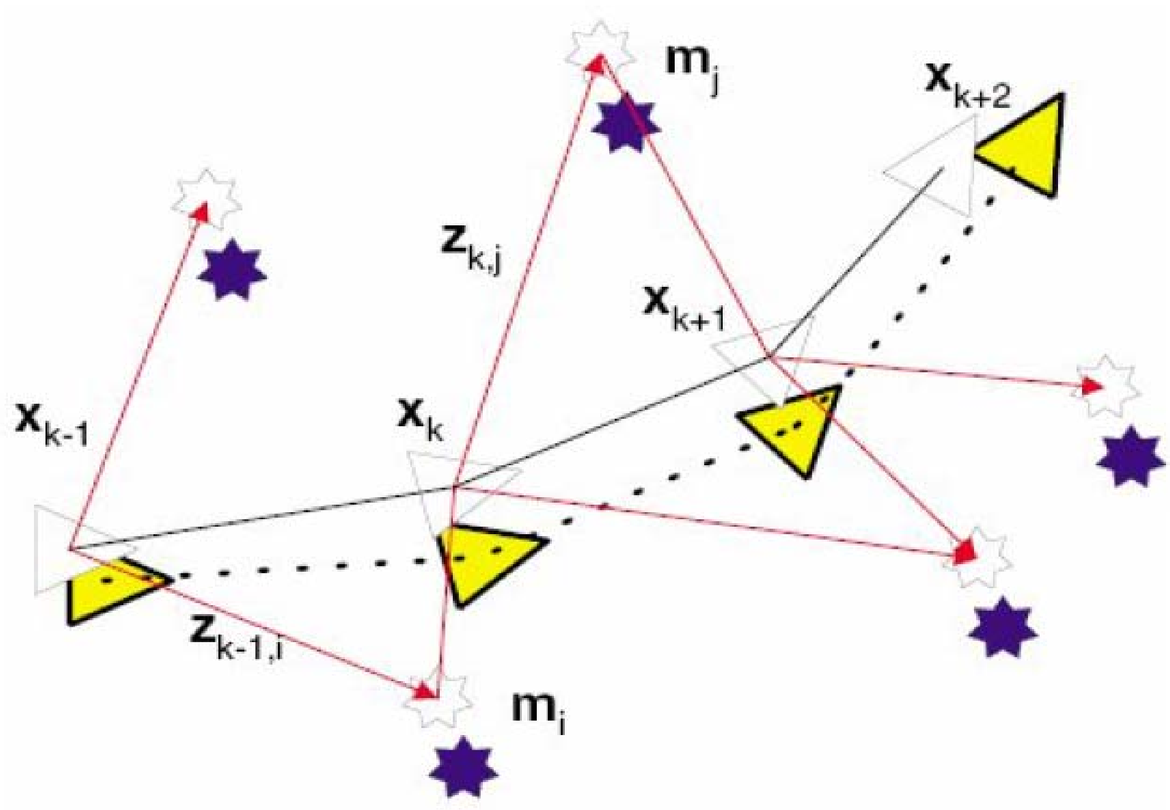
- From knowledge of observations Z^k
- Determine vehicle location X^k
- Build map m of landmark locations

- x_k: location of vehicle at time k
- u_k: a control vector applied at k-1 to drive the vehicle from x_{k-1} to x_k
- m_i: true location of i^{th} landmark
- z_k: observation of a landmark taken at time k
- X^k: history of states $\{x_1, x_2, x_3, \ldots, x_k\}$
- U^k: history of control inputs $\{u_1, u_2, u_3, \ldots, u_k\}$
- m: set of all landmarks
- Z^k: history of all observations $\{z_1, z_2, \ldots, z_k\}$



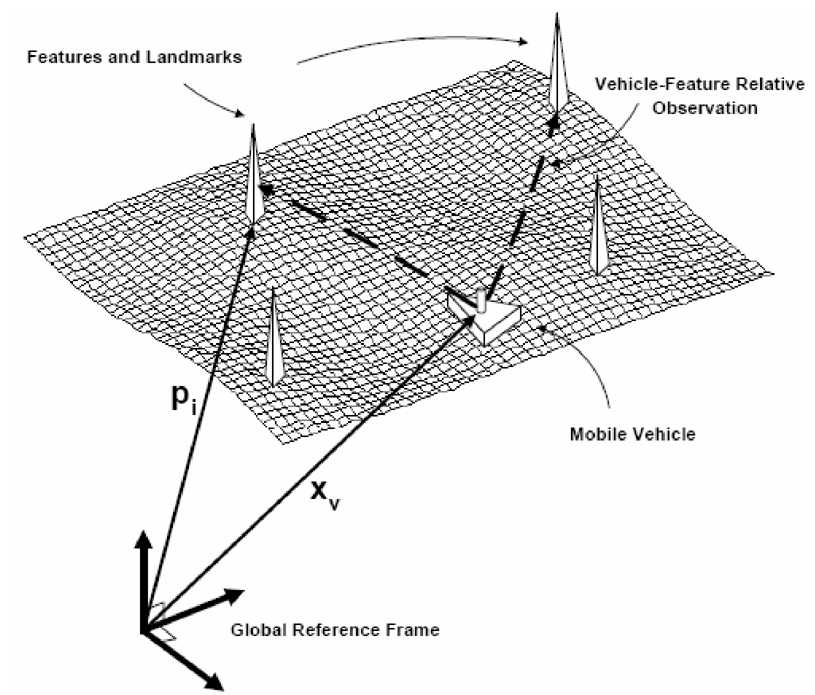
Simultaneous Localization and Mapping

- Localization and mapping are coupled problems
- A solution can only be obtained if the localization and mapping processes are considered together



SLAM Fundamentals

- A vehicle with a known kinematic model moving through an environment containing a population of landmarks (process model)
- The vehicle is equipped with a sensor that can take measurements of the relative location between any individual landmark and the vehicle itself (observation model)



Process Model

- For better understanding, a linear model of the vehicle is assumed
- If the state of the vehicle is given as $x_v(k)$ then the vehicle model is

$$x_v(k+1) = F_v(k)x_v(k) + u_v(k+1) + w_v(k+1)$$

- where
  - $F_v(k)$ is the state transition matrix
  - $u_v(k)$ is a vector of control inputs
  - $w_v(k)$ is a vector of uncorrelated process noise errors with zero mean and covariance $Q_v(k)$
- The state transition equation for the i^{th} landmark is

$$p_i(k+1) = p_i(k) = p_i$$

- SLAM considers all landmarks stationary

Process Model

- The augmented state vector containing both the state of the vehicle and the state of all landmark locations is

$$x(k) = \begin{bmatrix} x_v^T(k) & p_1^T & \dots & p_N^T \end{bmatrix}^T$$

- The state transition model for the complete system is now

$$\begin{bmatrix} x_v(k+1) \\ p_1 \\ \vdots \\ p_N \end{bmatrix} = \begin{bmatrix} F_v(k) & 0 & \dots & 0 \\ 0 & I_{p_1} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & I_{p_N} \end{bmatrix} \begin{bmatrix} x_v(k) \\ p_1 \\ \vdots \\ p_N \end{bmatrix} + \begin{bmatrix} u_v(k+1) \\ 0_{p_1} \\ \vdots \\ 0_{p_N} \end{bmatrix} + \begin{bmatrix} w_v(k+1) \\ 0_{p_1} \\ \vdots \\ 0_{p_N} \end{bmatrix}$$

- where
  - I_{pi} is the $\dim(p_i) \times \dim(p_i)$ identity matrix
  - 0_{pi} is the $\dim(p_i)$ null vector

Observation Model

- Assuming the observation to be linear, the observation model for the ith landmark is given as

$$z(k) = H_i x(k) + v_i(k)$$

- where
  - $v_i(k)$ is a vector of uncorrelated observation errors with zero mean and variance $R_i(k)$
  - H_i is the observation matrix that relates the sensor output $z_i(k)$ to the state vector $x(k)$ when observing the i^{th} landmark and is written as

$$H_i = [-H_v, 0 \dots 0, H_{pi}, 0 \dots 0]$$

- Re-expressing the observation model

$$z(k) = H_{pi} p - H_v x_v(k) + v_i(k)$$

Estimation Process

- Objective
  - The state of the discrete-time process x_k needs to be estimated based on the measurement z_k
  - This is the exact definition of the Kalman filter
- Kalman Filter
  - Recursively computes estimates of state x(k) which is evolving according to the process and observation models
  - The filter proceeds in three stages
    - Prediction
    - Observation
    - Update

Estimation Process

Prediction

- After initializing the filter (i.e. setting values for $\hat{x}$(k) and P(k)), a prediction is generated for
  - The a priori state estimate

$$\hat{x}(k+1|k) = F(k)\hat{x}(k|k) + u(k)$$

  - The a priori observation relative to the i^th landmark

$$\hat{z}_i(k+1|k) = H_i(k)\hat{x}(k+1|k)$$

  - The a priori state covariance (e.g. a measure of how uncertain the states computed by the process model are)

$$P(k+1|k) = F(k)P(k|k)F^T(k) + Q(k)$$

Estimation Process

Observation

- Following the prediction, an observation $z_i(k+1)$ of the i^{th} landmark is made using the observation model
- An innovation and innovation covariance matrix are calculated
  - Innovation is the discrepancy between the actual measurement z_k and the predicted measurement $\hat{z}(k)$

$$v_i(k+1) = z_i(k+1) - \hat{z}_i(k+1 \mid k)$$

$$S_i(k+1) = H_i(k)P(k+1 \mid k)H_i^T(k) + R_i(k+1)$$

Estimation Process

Update

- The state estimate and corresponding state estimate covariance are then updated according to

$$\hat{x}(k+1 \mid k+1) = \hat{x}(k+1 \mid k) + W_i(k+1)\, v_i(k+1)$$

$$P(k+1 \mid k+1) = P(k+1 \mid k) - W_i(k+1) S(k+1) W_i^T(k+1)$$

- where the gain matrix $W_i(k+1)$ is given by

$$W_i(k+1) = P(k+1 \mid k) H_i^T(k) S_i^{-1}(k+1)$$

Kalman Filter

- Developed by Rudolph E. Kalman in 1960
- A set of mathematical equations that provides an efficient computational (recursive) means to estimate the state of a process
- It supports estimations of
  - Past states
  - Present states
  - Future states
- and can do so when the nature of the modeled system is unknown!

Kalman Filter Properties

- Given all measurements up to current time, the Kalman filter algorithm is the optimal Minimum Mean Squared Error (MMSE) estimator of the state
- Provided that:
  - initial state is Gaussian with known mean and covariance;
  - process and observations models are linear;
  - and noise terms are uncorrelated, white, Gaussian, zero mean and with known covariances.

Discrete Kalman Filter

Process Model

- Assumes true state at time k evolves from state (k-1) according to

$$x(k) = F\,x(k-1) + G\,u(k-1) + w(k)$$

- where
  - F is the state transition model (A matrix)
  - G is the control input matrix (B matrix)
  - w(k) is the process noise which is assumed to be white and have a normal probability distribution

$$p(w) \sim N(0,Q)$$

Discrete Kalman Filter

Observation Model

- At time k, a measurement z(k) of the true state x(k) is made according to

$$z(k) = H\,x(k) + v(k)$$

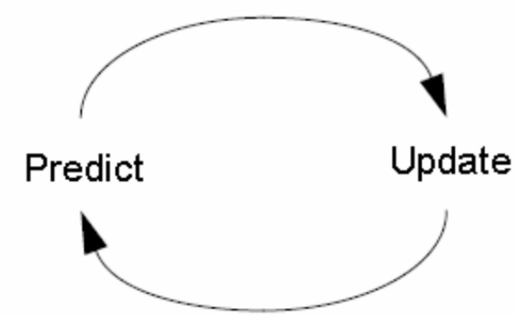
- where
  - H is the observation matrix and relates the measurement z(k) to the state vector x(k)
  - v(k) is the observation noise which is assumed to be white and have a normal probability distribution

$$p(w) \sim N(0,R)$$

Discrete Kalman Filter

Algorithm

- Recursive
  - Only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state
- The state of the filter is represented by two variables
  - x(k): estimate of the state at time k
  - P(k|k): error covariance matrix (a measure of the estimated accuracy of the state estimate)
- The filter has two distinct stages
  - Predict (and observe)
  - Update



Discrete Kalman Filter (Notation 1)

Prediction

- Predicted state $\boxed{\hat{x}(k \mid k-1)} = F(k)\hat{x}(k-1 \mid k-1) + B(k)u(k-1)$
- Predicted covariance $\boxed{P(k \mid k-1)} = F(k)P(k-1 \mid k-1)F(k)^T + Q(k)$

Observation

- Innovation $\tilde{y}(k) = z(k) - H(k)\hat{x}(k \mid k-1)$
- Innovation covariance $S(k) = H(k)P(k \mid k-1)H(k)^T + R(k)$

Update

Not the same variable!!

- Optimal Kalman gain $K(k) = P(k \mid k-1)H(k)^T S(k)^{-1}$
- Updated state $\boxed{\hat{x}(k \mid k)} = \hat{x}(k \mid k-1) + K(k)\tilde{y}(k)$
- Updated covariance $\boxed{P(k \mid k)} = (I - K(k)H(k))P(k \mid k-1)$

Not the same variable!!

Discrete Kalman Filter (Notation 2)

Prediction

- Predicted state $\hat{x}(k)^{-} = F(k)\hat{x}(k-1) + Bu(k-1)$
- Predicted estimate covariance $P(k)^{-} = FP(k-1)F^{T} + Q$

Observation

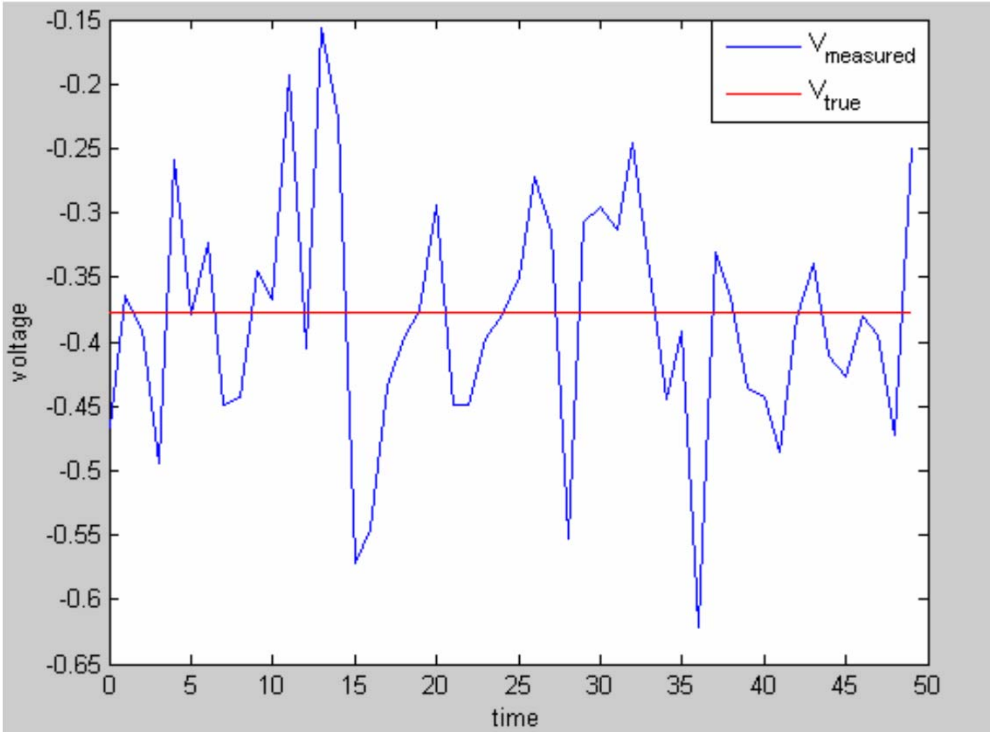
- Innovation $\tilde{y}(k) = z(k) - H\hat{x}(k)^{-}$
- Innovation covariance $S(k) = HP(k)^{-}H^{T} + R$

Update

- Optimal Kalman gain $K(k) = P(k)^{-}HS(k)^{-1}$
- Updated state estimate $\hat{x}(k) = \hat{x}(k)^{-} + K(k)\tilde{y}(k)$
- Updated estimate covariance $P(k) = (I - K(k)H)P(k)^{-}$

Kalman Filter Example

- Estimate a scalar random constant (e.g. voltage)
- \- Measurements are corrupted by 0.1 volt RMS white noise



Kalman Filter Example

Process Model

- Governed by the linear difference equation

$$x(k) = Fx(k-1) + Gu(k-1) + w(k)$$

$$x(k) = x(k-1) + w(k) \longrightarrow$$

State doesn't change (F=1)
No control input (u=0)

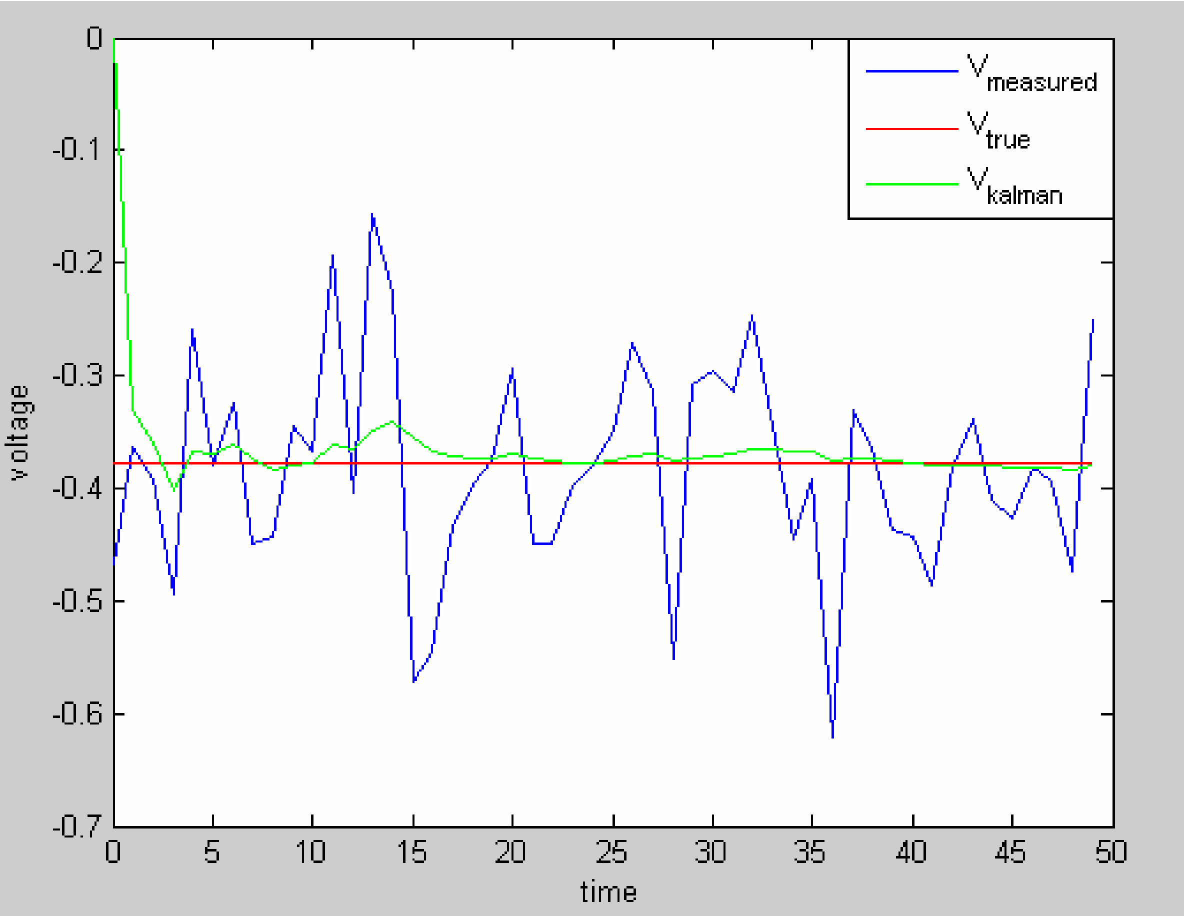
- with a measurement

$$z(k) = Hx(k) + v(k)$$

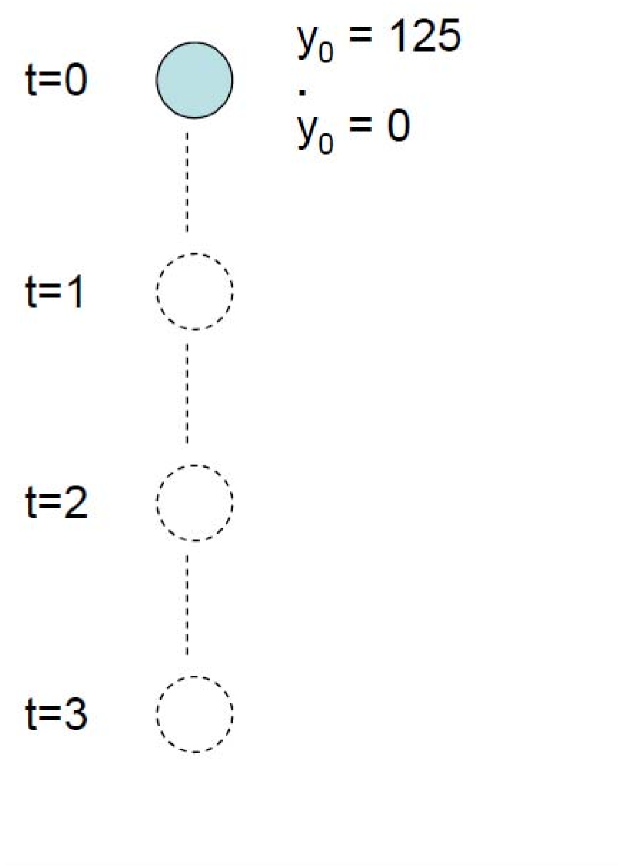
$$z(k) = x(k) + v(k) \longrightarrow$$

Measurement is of state directly (H=1)

Kalman Filter Example



Another Example

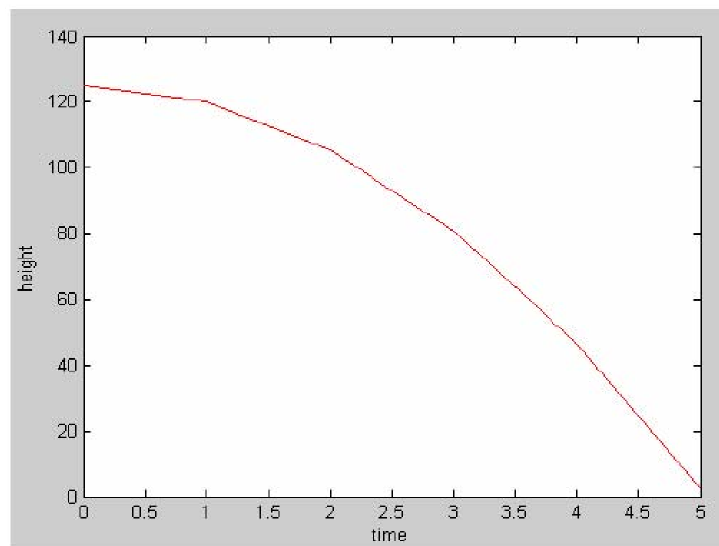


Kinematic Equations

$$y - y_0 = \dot{y}_0 \Delta t + \frac{1}{2} a (\Delta t)^2$$

$$\dot{y} = \dot{y}_0 + a \Delta t$$

Position (from model)



Process Model

$$y(k+1) = y(k) + \dot{y}(k)\Delta t + \frac{1}{2}a(\Delta t)^2$$

$$\dot{y}(k+1) = \dot{y}(k) + a\Delta t$$

where $\begin{bmatrix} y(k+1) \\ \dot{y}(k+1) \end{bmatrix} = x(k+1)$ and $\begin{bmatrix} y(k) \\ \dot{y}(k) \end{bmatrix} = x(k)$

so

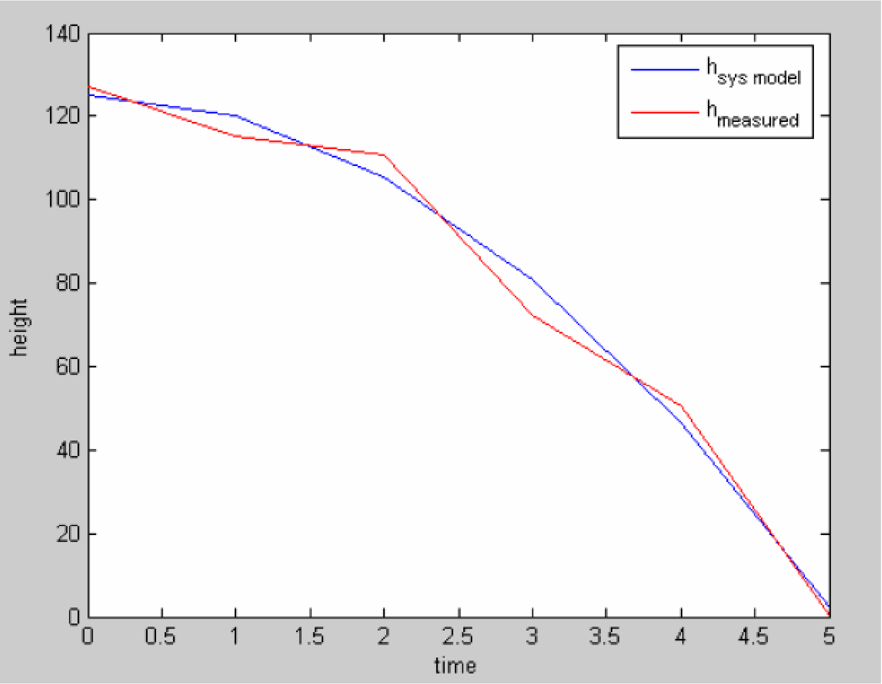
$$x(k+1) = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} \dfrac{\Delta t^2}{2} \\ \Delta t \end{bmatrix} a$$

Observation Model

$$z(k) = Hx(k) + v(k)$$

where $H = [1 \quad 0]$ because z is a measurement of the height directly

z = [127.0	115.3	110.9
72.4	50.7	0.3]



Kalman Filter

$$R = 1$$
$$\Delta t = 1$$

Initial Estimates

$$\hat{x}(k-1) = \begin{bmatrix} y(k-1) \\ \dot{y}(k-1) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \qquad P(k-1) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Prediction

$$\hat{x}(k)^{-} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \hat{x}(k-1) + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} * -9.81 \qquad P(k)^{-} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P(k-1) \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

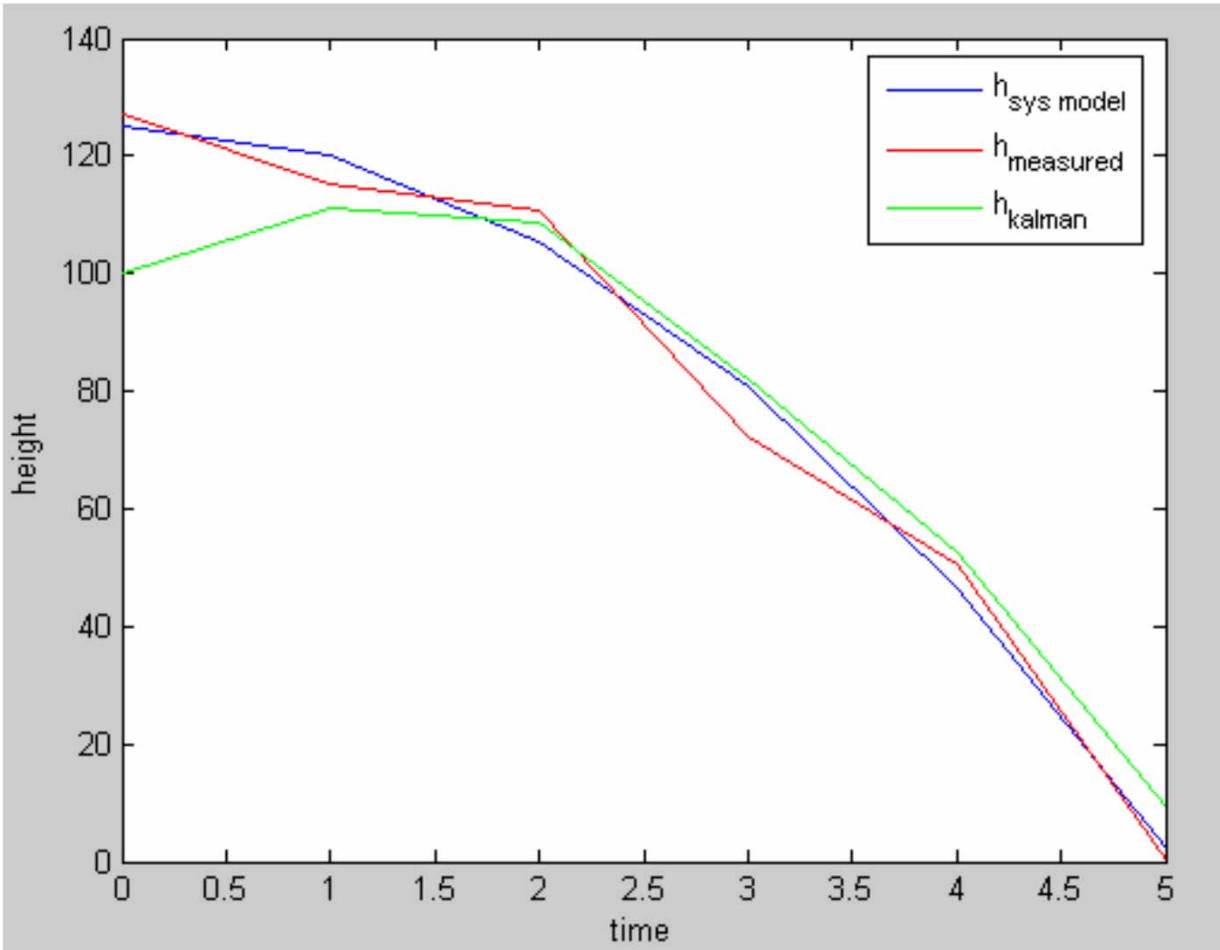
Observation and Update

$$K(k) = P(k)^{-} H^{T} (HP(k)^{-} H^{T} + R)^{-1}$$

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)[z(k) - H\hat{x}(k)^{-}]$$

$$P(k) = (I - K(k)H)P(k)^{-}$$

Kalman Filter



Non-Linear Systems

Kalman Filter

- Limited to linear systems
- A non-linearity in a system can be associated with either the process model or the observation model (or both)

Extended Kalman Filter

- Process and observation models can both be non-linear

$$x(k) = f(x(k-1),u(k-1),w(k-1))$$

$$z(k) = h(x(k),v(k))$$

- where f and h are non-linear functions

Extended Kalman Filter

Noise Parameters

- In practice, one does not know the noise values w(k) and v(k) at every time step
- Instead, the state and measurement vector are approximated without them

$$\widetilde{x}(k) = f\left(\hat{x}(k-1), u(k), 0\right)$$

$$\widetilde{z}(k) = h\left(\widetilde{x}(k), 0\right)$$

- where $\hat{x}$(k) is some a posteriori estimate of the state

EKF

$$\tilde{x}(k) = f\left(\hat{x}(k-1), u(k), 0\right)$$
$$\tilde{z}(k) = h\left(\tilde{x}(k), 0\right)$$

To estimate a non-linear process, we need to linearize system at the current state

$$x(k) = \tilde{x}(k) + A\left(x(k-1) - \hat{x}(k-1)\right) + Ww(k-1)$$
$$z(k) = \tilde{z}(k) + J_h\left(x(k) - \tilde{x}(k)\right) + Vv(k)$$

$x(k), z(k)$	: actual state and measurement vectors
$\tilde{x}(k), \tilde{z}(k)$	: approximate state and measurement vectors
$\hat{x}(k)$	: a posteriori estimate of the state at step k
$w(k), v(k)$	: process and measurement noise
A	: Jacobian matrix of partial derivatives of f w.r.t. x
W	: Jacobian matrix of partial derivatives of f w.r.t. w
J_h	: Jacobian matrix of partial derivatives of h w.r.t. x
V	: Jacobian matrix of partial derivatives of h w.r.t. v

EKF

Let's define new notations for the prediction and measurement error

$$\tilde{e}_x(k) = x(k) - \tilde{x}(k) \qquad \tilde{e}_z(k) = z(k) - \tilde{z}(k)$$

Therefore, we have

$$\tilde{e}_x(k) \approx A\big(x(k-1) - \hat{x}(k-1)\big) + \varepsilon(k)$$

$$\tilde{e}_z(k) \approx J_h \tilde{e}_x(k) + \eta(k)$$

where ε(k) and η(k) represent new noise var.

$$\boxed{\begin{array}{l} p(\varepsilon(k)) \sim N\left(0, WQ(k)W^T\right) \\ p(\eta(k)) \sim N\left(0, VR(k)V^T\right) \end{array}}$$

The above equations are linear and closely resemble the difference equations from the discrete KF. Therefore, we could use a 2nd Kalman filter to estimate the prediction error

$$\hat{e}(k) = e(k)^- + K_k\big(z(k) - \tilde{z}(k)\big) = K_k \tilde{e}_z(k)$$ **(update equation)**

$$\hat{e}_x(k) = \boxed{\hat{x}(k)} - \tilde{x}(k)$$

This is what we are trying to find!!

EKF

- Rearranging the predicted error estimate yields

$$\hat{e}_x(k) = \hat{x}(k) - \tilde{x}(k) \quad \Longrightarrow \quad \hat{x}(k) = \tilde{x}(k) + \hat{e}_x(k)$$

- Plugging in from the previous slide

$$\hat{x}(k) = \tilde{x}(k) + K_k \tilde{e}_z(k) \quad \Longrightarrow \quad \boxed{\hat{x}(k) = \tilde{x}(k) + K_k\big(z(k) - \tilde{z}(k)\big)}$$

- The equation above can now be used in the measurement update in the EKF

EKF

Prediction

- Predicted state $\hat{x}(k)^{-} = f\left(\hat{x}(k-1), u(k-1), 0\right)$
- Predicted estimate covariance $P(k)^{-} = F(k)P(k-1)F(k)^{T} + W(k)Q(k-1)W(k)^{T}$

Observation

- Innovation $\tilde{y}(k) = z(k) - H$ where H is the sensor model
- Innovation covariance $S(k) = J_{h}(k)P(k)^{-}J_{h}(k)^{T} + V(k)R(k)V(k)^{T}$

Update

- Optimal Kalman gain $K(k) = P(k)^{-}J_{h}(k)^{T}S(k)^{-1}$
- Updated state estimate $\hat{x}(k) = \hat{x}(k)^{-} + K(k)\tilde{y}(k)$
- Updated estimate covariance $P(k) = \left(I - K(k)J_{h}(k)\right)P(k)^{-}$

EKF

Prediction

(1) Project the state ahead

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

(2) Project the error covariance ahead

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

Observation and Update

(1) Compute the Kalman gain

$$K(k) = P(k)^- J_h(k)^T \left(J_h(k)P(k)^- J_h(k)^T + V(k)R(k)V(k)^T\right)$$

(2) Update estimate with measurement z(k)

$$\hat{x}(k) = \hat{x}(k)^- + K(k)\left[z(k) - H\right]$$

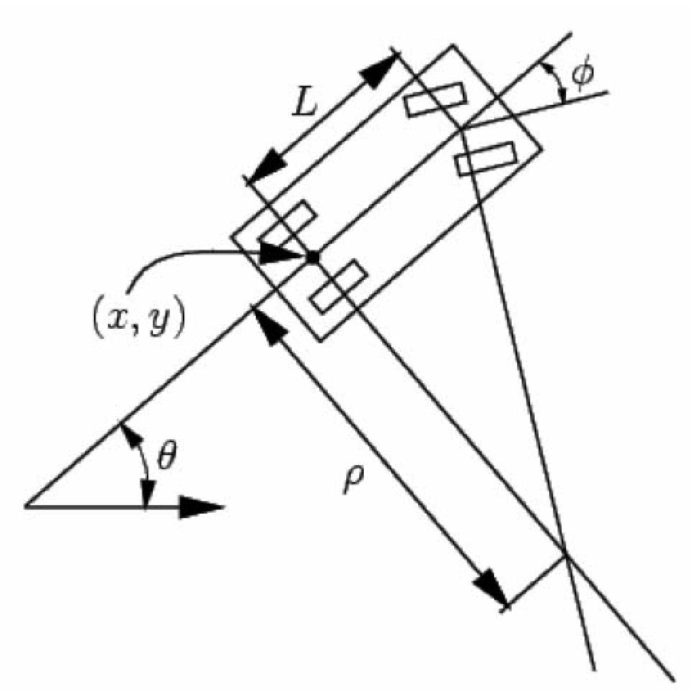
(3) Update error covariance

$$P(k) = \left(I - K(k)J_h(k)\right)P(k)^-$$

Initial estimates for

$\hat{x}(k-1)$ & $P(k-1)$

Simple Robot Model



Kinematic Equations

$$\left.\begin{aligned}\dot{x} &= V\cos\theta \\ \dot{y} &= V\sin\theta \\ \dot{\theta} &= \frac{V\tan\phi}{L}\end{aligned}\right\}$$

Non-linear!

Simple Robot Model

Kinematic Equations

$$\dot{x} = V\cos\theta$$

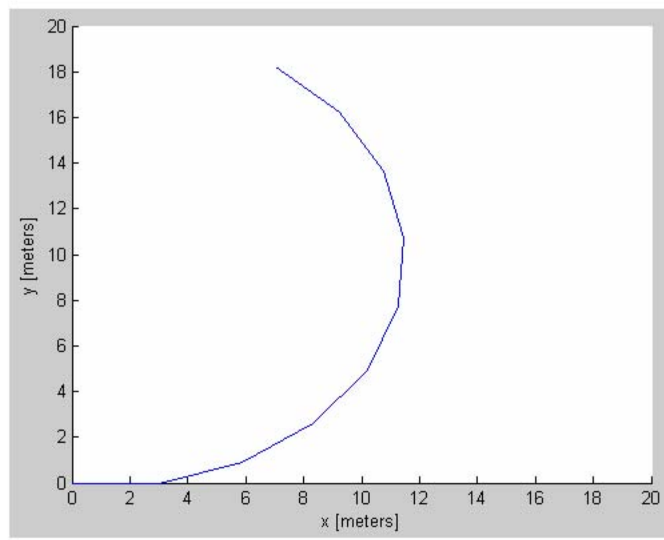
$$\dot{y} = V\sin\theta$$

$$\dot{\theta} = \frac{V\tan\phi}{L}$$

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \theta(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} x(k) + \Delta t V(k)\cos\theta(k) \\ y(k) + \Delta t V(k)\sin\theta(k) \\ \theta(k) + \dfrac{\Delta t V(k)\tan\phi(k)}{L} \end{bmatrix}}^{f(x,u,w)}$$

Assumptions

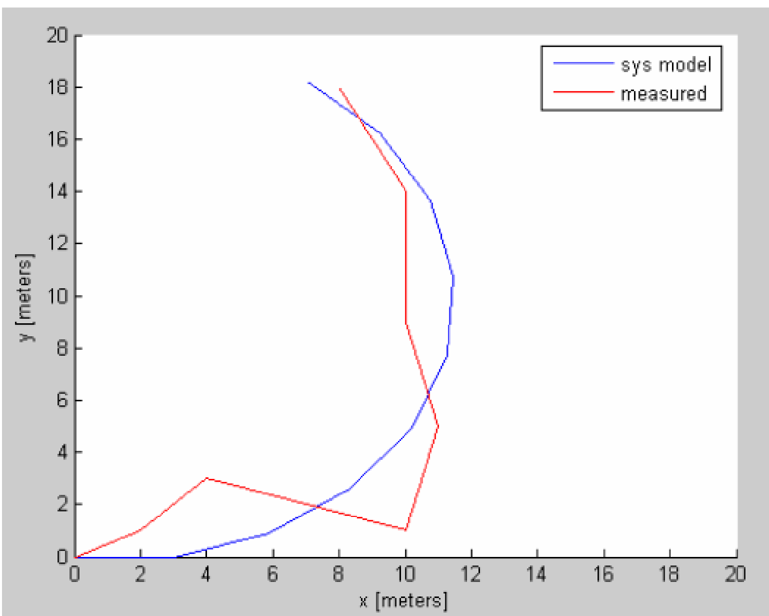
- System inputs
  - Velocity (assumed constant, vel=3)
  - Steering angle (ϕ)
- Δt is fixed and equal to 1
- L=1
- 10 iterations (N=10)

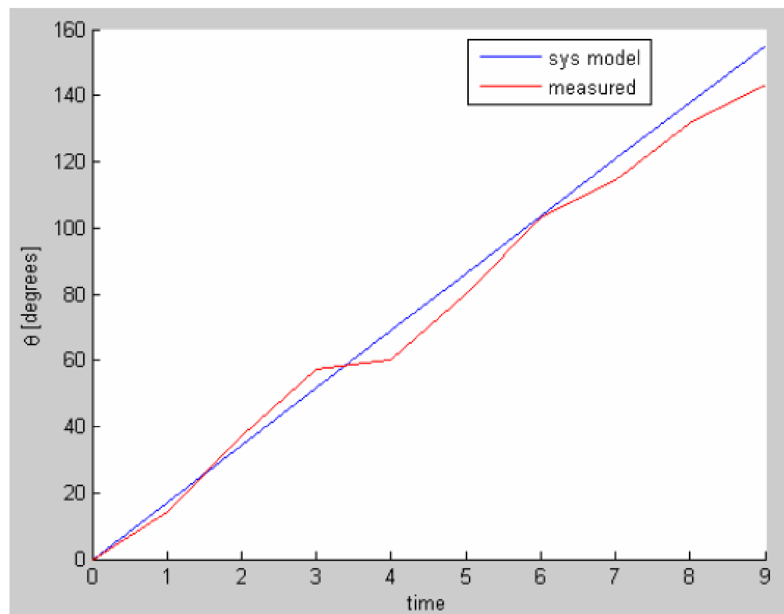


Observation Model

Measurements are taken from an overhead camera, and thus x, y, and θ can be measured directly

$$z(k) = h\big(x(k), v(k)\big) \quad \Rightarrow \quad z(k) = \begin{bmatrix} x(k) + v_x \\ y(k) + v_y \\ \theta(k) + v_\theta \end{bmatrix}$$





EKF

Prediction

$$\hat{x}(k)^{-} = f\left(\hat{x}(k-1), u(k-1), 0\right)$$ from robot model

$$P(k)^{-} = \underline{F(k)}P(k-1)\underline{F(k)^{T}} + \underline{W(k)}Q(k-1)\underline{W(k)^{T}}$$

$$x(k+1) = f\big(x(k), u(k), w(k)\big) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\theta(k) \\ y(k) + \Delta t V(k)\sin\theta(k) \\ \theta(k) + \dfrac{\Delta t V(k)\tan\phi(k)}{L} \end{bmatrix}$$

Need to calculate Jacobians!

$$F(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial \theta} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial \theta} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -V\sin\theta \\ 0 & 1 & V\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$W(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial w_x} & \dfrac{\partial f_1}{\partial w_y} & \dfrac{\partial f_1}{\partial w_\theta} \\ \dfrac{\partial f_2}{\partial w_x} & \dfrac{\partial f_2}{\partial w_y} & \dfrac{\partial f_2}{\partial w_\theta} \\ \dfrac{\partial f_3}{\partial w_x} & \dfrac{\partial f_3}{\partial w_y} & \dfrac{\partial f_3}{\partial w_\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

EKF

Kalman Gain

$$K(k) = P(k)^{-} \underline{J_h(k)}^T \left(\underline{J_h(k)} P(k)^{-} \underline{J_h(k)}^T + \underline{V(k)} R(k) \underline{V(k)}^T\right)^{-1}$$

$$z(k) = h\left(x(k), v(k)\right) = \begin{bmatrix} x(k) + v_x \\ y(k) + v_y \\ \theta(k) + v_\theta \end{bmatrix}$$

Need to calculate Jacobians!

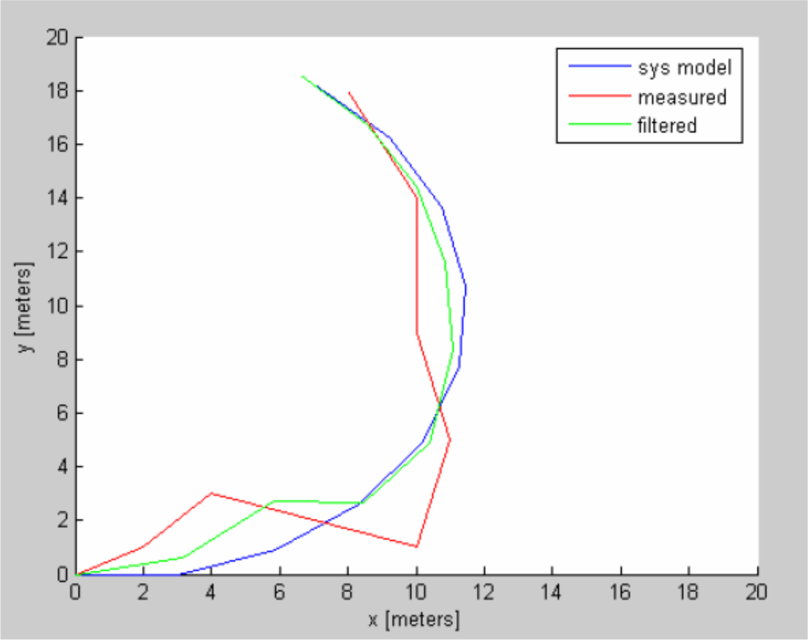
$$J_h(k) = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial \theta} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial \theta} \\ \frac{\partial h_3}{\partial x} & \frac{\partial h_3}{\partial y} & \frac{\partial h_3}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad V(k) = \begin{bmatrix} \frac{\partial h_1}{\partial v_x} & \frac{\partial h_1}{\partial v_y} & \frac{\partial h_1}{\partial v_\theta} \\ \frac{\partial h_2}{\partial v_x} & \frac{\partial h_2}{\partial v_y} & \frac{\partial h_2}{\partial v_\theta} \\ \frac{\partial h_3}{\partial v_x} & \frac{\partial h_3}{\partial v_y} & \frac{\partial h_3}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

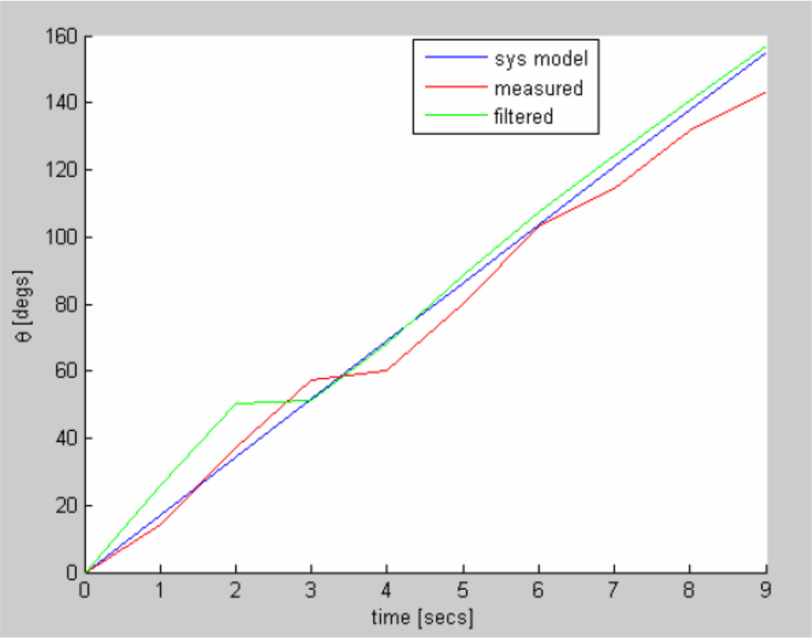
EKF

Measurement Update

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\left(z(k) - H\right)$$

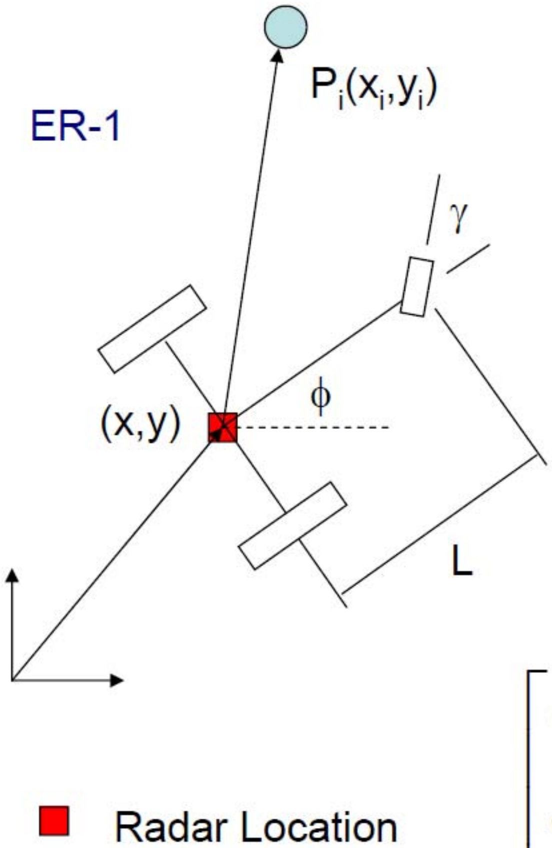
$$P(k) = \left(I - K(k)J_h(k)\right)P(k)^{-}$$





SLAM Example - Single Landmark

Robot Process Model



Kinematic Equations

$$\left.\begin{aligned} \dot{x} &= V\cos\varphi \\ \dot{y} &= V\sin\varphi \\ \dot{\varphi} &= \frac{V\tan\gamma}{L} \end{aligned}\right\}$$

Non-linear!

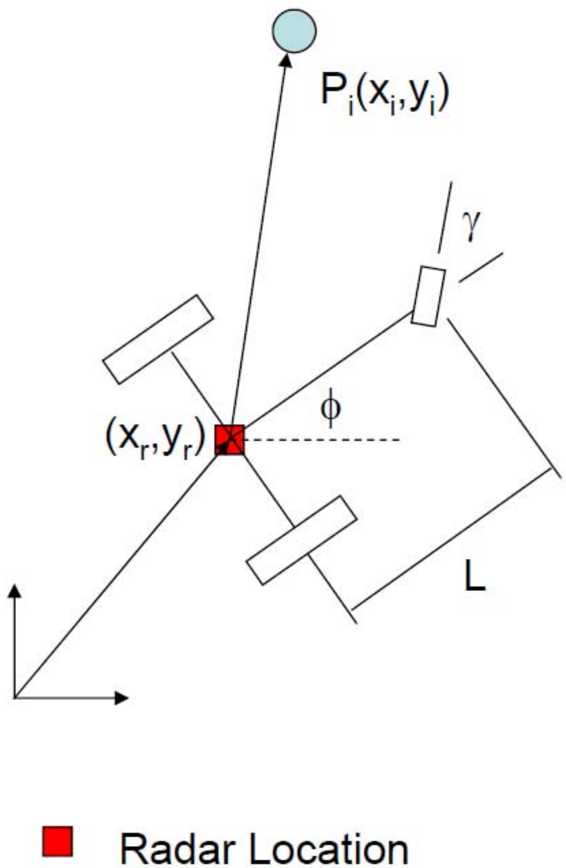
f(x,u,w)

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \end{bmatrix} + w(k)$$

Objective

- Based on system inputs, V and γ (with sensor feedback, i.e. optical encoders) at time k, estimate the vehicle position at time (k+1)

Landmark Process Model



Recall that in the SLAM algorithm, landmarks are assumed to be stationary. Therefore,

$$p_i(k+1) = p_i(k)$$

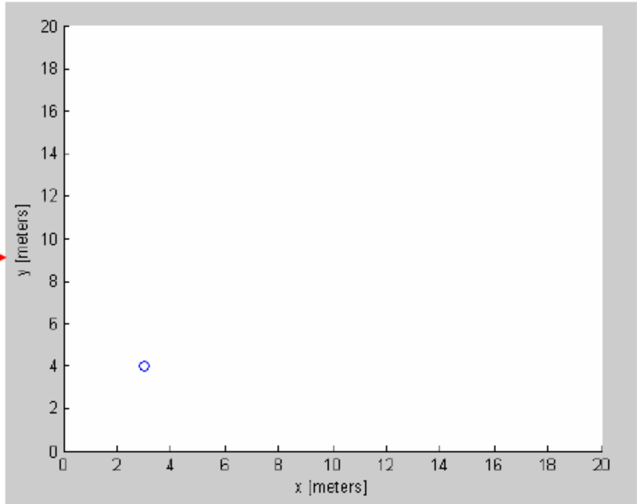
$$\begin{bmatrix} x_i(k+1) \\ y_i(k+1) \end{bmatrix} = \begin{bmatrix} x_i(k) \\ y_i(k) \end{bmatrix}$$

$$\begin{bmatrix} x_1(k+1) \\ y_1(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ y_1(k) \end{bmatrix}$$

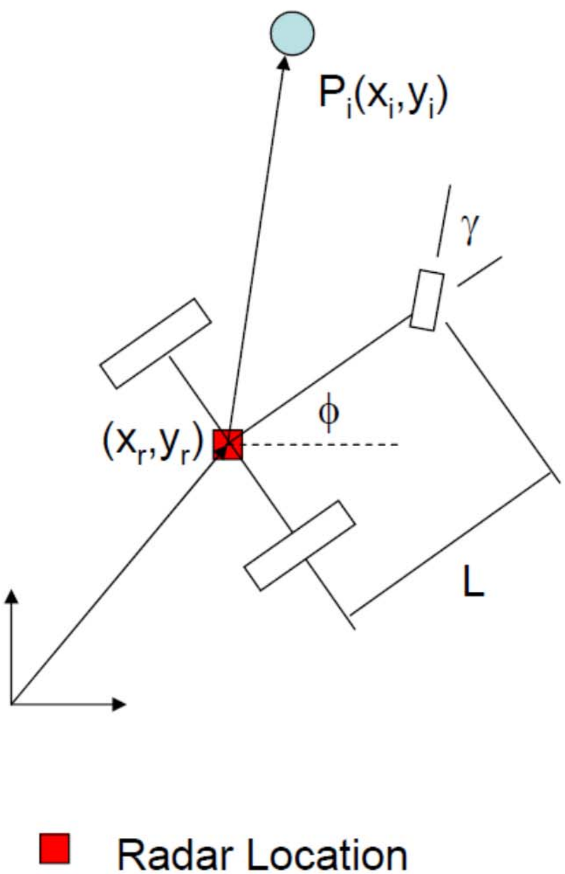
Overall System Process Model

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \\ x_1(k+1) \\ y_1(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k) \cos\varphi(k) \\ y(k) + \Delta t V(k) \sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k) \tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

Landmark 1: (3,4)



Observation Model



$$z(k) = h\big(x(k), v(k)\big)$$

The radar used in the experiment returns the range $r_i(k)$ and bearing $\theta_i(k)$ to a landmark i. Thus, the observation model is

$$r_i(k) = \sqrt{\big(x_i - x_r(k)\big)^2 + \big(y_i - y_r(k)\big)^2} + v_r(k)$$

$$\theta_i(k) = \arctan\left(\frac{y_i - y_r(k)}{x_i - x_r(k)}\right) - \varphi(k) + v_\theta(k)$$

The Estimation Process

Prediction

$$\hat{x}(k)^{-} = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$P(k)^{-} = F(k)P(k-1)F(k)^{T} + W(k)Q(k-1)W(k)^{T}$$

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$F(k) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \varphi} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \varphi} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \varphi} & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial y_1} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial \varphi} & \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial y_1} \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial y} & \frac{\partial f_5}{\partial \varphi} & \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial y_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta t V(k)\sin\varphi(k) & 0 & 0 \\ 0 & 1 & \Delta t V(k)\cos\varphi(k) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Estimation Process

Prediction

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

$$W(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial w_x} & \dfrac{\partial f_1}{\partial w_y} & \dfrac{\partial f_1}{\partial w_\varphi} & \dfrac{\partial f_1}{\partial w_{x_1}} & \dfrac{\partial f_1}{\partial w_{y_1}} \\ \dfrac{\partial f_2}{\partial w_x} & \dfrac{\partial f_2}{\partial w_y} & \dfrac{\partial f_2}{\partial w_\varphi} & \dfrac{\partial f_1}{\partial w_{x_1}} & \dfrac{\partial f_1}{\partial w_{y_1}} \\ \dfrac{\partial f_3}{\partial w_x} & \dfrac{\partial f_3}{\partial w_y} & \dfrac{\partial f_3}{\partial w_\varphi} & \dfrac{\partial f_1}{\partial w_{x_1}} & \dfrac{\partial f_1}{\partial w_{y_1}} \\ \dfrac{\partial f_4}{\partial w_x} & \dfrac{\partial f_4}{\partial w_y} & \dfrac{\partial f_4}{\partial w_\varphi} & \dfrac{\partial f_4}{\partial w_{x_1}} & \dfrac{\partial f_4}{\partial w_{y_1}} \\ \dfrac{\partial f_5}{\partial w_x} & \dfrac{\partial f_5}{\partial w_y} & \dfrac{\partial f_5}{\partial w_\varphi} & \dfrac{\partial f_5}{\partial w_{x_1}} & \dfrac{\partial f_5}{\partial w_{y_1}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Estimation Process

Kalman Gain

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

$$z(k) = \begin{bmatrix} r_i(k) \\ \theta_i(k) \end{bmatrix} = \begin{bmatrix} \sqrt{\left(x_i - \hat{x}(k)^{-}\right)^2 + \left(y_i - \hat{y}(k)^{-}\right)^2} \\ \tan^{-1}\left(\dfrac{y_i - \hat{y}(k)^{-}}{x_i - \hat{x}(k)^{-}} \right) - \hat{\varphi}(k)^{-} \end{bmatrix} + v(k)$$

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial \varphi} & \dfrac{\partial h_1}{\partial x_1} & \dfrac{\partial h_1}{\partial y_1} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} & \dfrac{\partial h_2}{\partial \varphi} & \dfrac{\partial h_2}{\partial x_1} & \dfrac{\partial h_2}{\partial y_1} \end{bmatrix} = \begin{bmatrix} \dfrac{x - x_i}{r} & \dfrac{y - y_i}{r} & 0 & \dfrac{x_i - x}{r} & \dfrac{y_i - y}{r} \\ \dfrac{y_i - y}{r^2} & \dfrac{x - x_i}{r^2} & -1 & \dfrac{y - y_i}{r^2} & \dfrac{x_i - x}{r^2} \end{bmatrix}$$

where $r = \sqrt{(x_i - x)^2 + (y_i - y)^2}$

The Estimation Process

Kalman Gain

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

$$z(k) = \begin{bmatrix} r_i(k) \\ \theta_i(k) \end{bmatrix} = \begin{bmatrix} \sqrt{\left(x_i - \hat{x}(k)^{-}\right)^2 + \left(y_i - \hat{y}(k)^{-}\right)^2} \\ \tan^{-1}\left(\dfrac{y_i - \hat{y}(k)^{-}}{x_i - \hat{x}(k)^{-}} \right) - \hat{\varphi}(k)^{-} \end{bmatrix} + v(k)$$

$$V(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial v_r} & \dfrac{\partial h_1}{\partial v_\theta} \\ \dfrac{\partial h_2}{\partial v_r} & \dfrac{\partial h_2}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Estimation Process

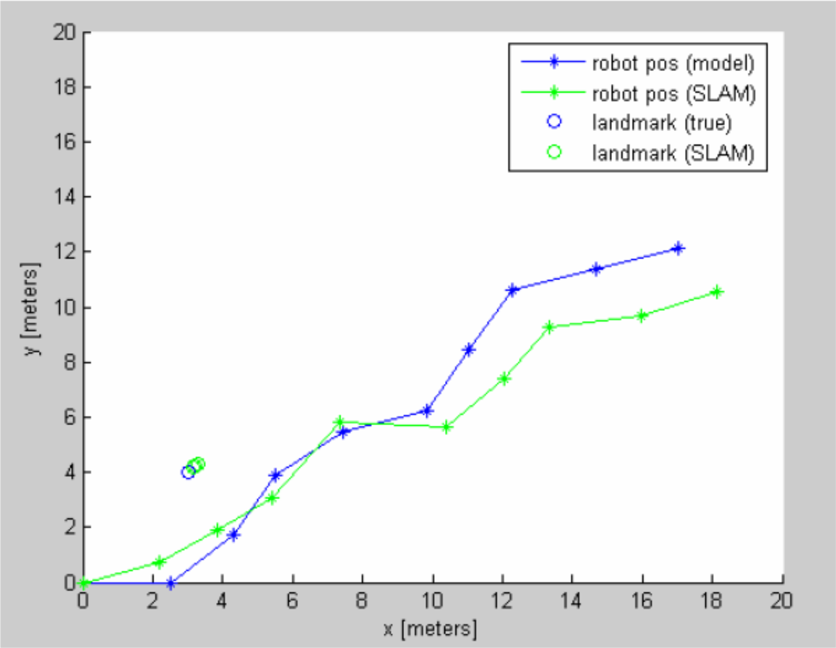
Measurement Update

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\big(\underbrace{z(k) - H(k)}_{\text{Innovation}}\big)$$

$$P(k) = \big(I - K(k)J_h(k)\big)P(k)^{-}$$

z(k) is 10 fabricated measurements of range and bearing to landmark 1.

There is only one landmark and it is incorporated into the model from the start.

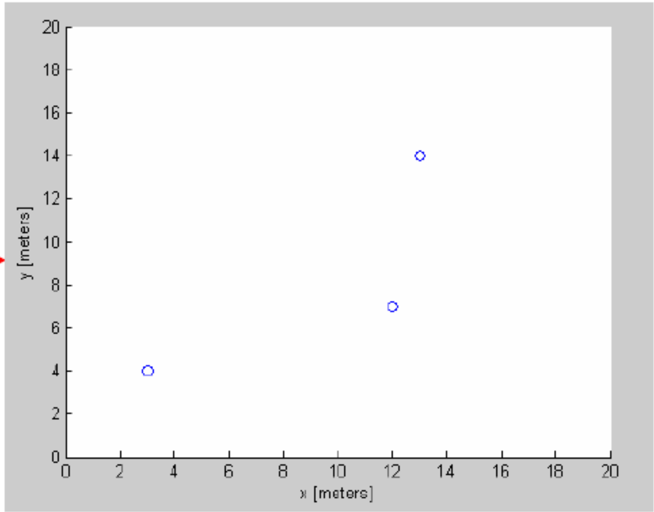


SLAM Example – Multiple Landmarks

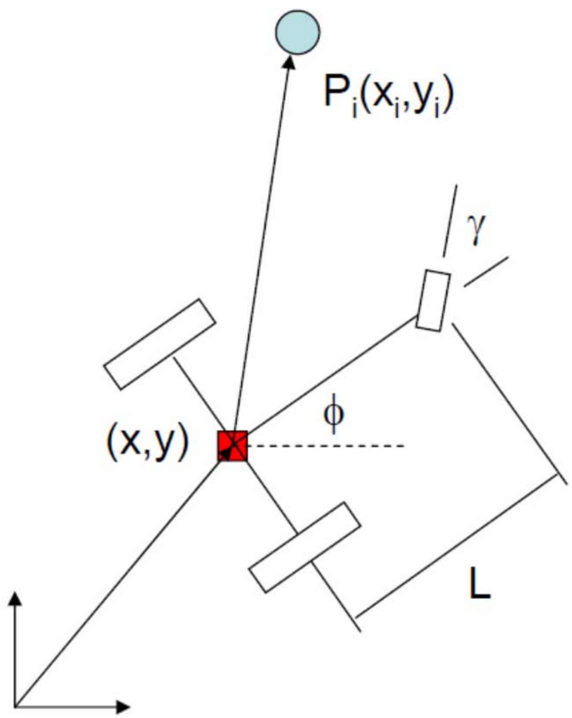
Overall System Process Model

$$
\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \\ p_1(k+1) \\ \vdots \\ p_N(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k) \cos\varphi(k) \\ y(k) + \Delta t V(k) \sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k) \tan\gamma(k)}{L} \\ p_1(k) \\ \vdots \\ p_N(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Landmark 1: (3,4)
Landmark 2: (12,7)
Landmark 3: (13,14)



Observation Model



Radar Location

$$z(k) = h\big(x(k), v(k)\big)$$

The radar used in the experiment returns the range $r_i(k)$ and bearing $\theta_i(k)$ to a landmark i. Thus, the observation model is

$$r_i(k) = \sqrt{\big(x_i - x_r(k)\big)^2 + \big(y_i - y_r(k)\big)^2} + v_r(k)$$

$$\theta_i(k) = \arctan\left(\frac{y_i - y_r(k)}{x_i - x_r(k)}\right) - \varphi(k) + v_\theta(k)$$

The Estimation Process

Prediction

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \end{bmatrix} + w(k)$$

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

Initially, before landmarks are added

$$F(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial \varphi} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial \varphi} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\Delta t V(k)\sin\varphi(k) \\ 0 & 1 & \Delta t V(k)\cos\varphi(k) \\ 0 & 0 & 1 \end{bmatrix}$$

$$W(k) = \begin{bmatrix} \dfrac{\partial f_1}{\partial w_x} & \dfrac{\partial f_1}{\partial w_y} & \dfrac{\partial f_1}{\partial w_\varphi} \\ \dfrac{\partial f_2}{\partial w_x} & \dfrac{\partial f_2}{\partial w_y} & \dfrac{\partial f_2}{\partial w_\varphi} \\ \dfrac{\partial f_3}{\partial w_x} & \dfrac{\partial f_3}{\partial w_y} & \dfrac{\partial f_3}{\partial w_\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Estimation Process

Kalman Gain

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

$$z(k) = \begin{bmatrix} r_i(k) \\ \theta_i(k) \end{bmatrix} = \begin{bmatrix} \sqrt{\left(x_i - \hat{x}(k)^{-}\right)^2 + \left(y_i - \hat{y}(k)^{-}\right)^2} \\ \tan^{-1}\left(\dfrac{y_i - \hat{y}(k)^{-}}{x_i - \hat{x}(k)^{-}} \right) - \hat{\varphi}(k)^{-} \end{bmatrix} + v(k)$$

Initially, before landmarks are added

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial x} & \dfrac{\partial h_1}{\partial y} & \dfrac{\partial h_1}{\partial \varphi} \\ \dfrac{\partial h_2}{\partial x} & \dfrac{\partial h_2}{\partial y} & \dfrac{\partial h_2}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \dfrac{x - x_i}{r} & \dfrac{y - y_i}{r} & 0 \\ \dfrac{y_i - y}{r^2} & \dfrac{x - x_i}{r^2} & -1 \end{bmatrix} \qquad V(k) = \begin{bmatrix} \dfrac{\partial h_1}{\partial v_r} & \dfrac{\partial h_1}{\partial v_\theta} \\ \dfrac{\partial h_2}{\partial v_r} & \dfrac{\partial h_2}{\partial v_\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $r = \sqrt{(x_i - x)^2 + (y_i - y)^2}$

The Estimation Process

Measurement Update

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\big(z(k) - H(k)\big)$$

$$P(k) = \big(I - K(k)J_h(k)\big)P(k)^{-}$$

Now, if a landmark is observed at t(k+1), the state model is updated

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \varphi(k+1) \\ x_1(k+1) \\ y_1(k+1) \end{bmatrix} = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$x_1(k+1) = x(k) + r\cos\theta \qquad y_1(k+1) = y(k) + r\sin\theta$$

The Estimation Process

Prediction (2)

$$\hat{x}(k)^- = f\left(\hat{x}(k-1), u(k-1), 0\right)$$

$$P(k)^- = F(k)P(k-1)F(k)^T + W(k)Q(k-1)W(k)^T$$

$$x(k+1) = \begin{bmatrix} x(k) + \Delta t V(k)\cos\varphi(k) \\ y(k) + \Delta t V(k)\sin\varphi(k) \\ \varphi(k) + \dfrac{\Delta t V(k)\tan\gamma(k)}{L} \\ x_1(k) \\ y_1(k) \end{bmatrix} + \begin{bmatrix} w_x(k) \\ w_y(k) \\ w_\varphi(k) \\ 0 \\ 0 \end{bmatrix}$$

$$F(k) = \begin{bmatrix} \dfrac{\partial f}{\partial(x, y, \varphi)} & 0 \\ 0 & I^{2N\ x\ 2N} \end{bmatrix}$$

where N is the number of landmarks

The Estimation Process

Kalman Gain (2)

$$K(k) = P(k)^{-} J_h(k)^T \left(J_h(k) P(k)^{-} J_h(k)^T + V(k) R(k) V(k)^T \right)^{-1}$$

If observing the 1st landmark

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h}{\partial(x, y, \varphi)} & \dfrac{\partial h}{\partial(x_i, y_i)} & 0 & \dots & 0 \end{bmatrix}$$

If observing the 2nd landmark

$$J_h(k) = \begin{bmatrix} \dfrac{\partial h}{\partial(x, y, \varphi)} & 0 & \dfrac{\partial h}{\partial(x_i, y_i)} & 0 & \dots & 0 \end{bmatrix}$$

Must repeat for each landmark!!

The Estimation Process

Measurement Update (2)

$$\hat{x}(k) = \hat{x}(k)^{-} + K(k)\big(z(k) - H(k)\big)$$

$$P(k) = \big(I - K(k)J_{h}(k)\big)P(k)^{-}$$